1995: PART A

1) $f(x) = \frac{2x}{\sqrt{x^2+x+1}}$

a) $x^2+x+1>0$
$x^2+x+(1/2)^2+1>0$
$(x+1/2)^2+1-1/4>0$
$(x+1/2)^2+3/4>0$ for all real x

Domain: All real x

b) see graph

c) $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x+1}} = \lim_{x\to\infty} \frac{\frac{2x}{x}}{\sqrt{\frac{x^2+x+1}{x^2}}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{2}{\sqrt{1+0+0}} = 2$

$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+x+1}} = \lim_{x\to-\infty} \frac{2}{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{2}{-\sqrt{1+0+0}} = -2$

$y=2, y=-2$

d) $f'(x) = \frac{x+2}{(x^2+x+1)^{3/2}}$
$f'(x)=0$

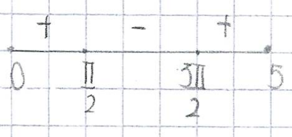
$x+2=0$
$x=-2$

$f(-2) = -\frac{4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$

Range: $\{y: -4\sqrt{3}/3 \le y < 2\}$

2) $v(t) = t\cos t,\ t \ge 0,\ y(0)=3$

a) $t\cos t > 0,\quad 0 \le t \le 5$
$t\cos t = 0$
$t=0 \quad \cos t = 0$
$\quad t = \frac{\pi}{2}, \frac{3\pi}{2}$



$(0, \pi/2), (3\pi/2, 5)$

b) $A(t) = v'(t)$
$A(t) = \cos t + t(-\sin t)$
$= \cos t - t\sin t$

c) $y(t) = \int v(t)\,dt$

$y(t) = \int t\cos t\,dt$ let $u=t \quad \frac{dv}{dt}=\cos t$
$\frac{du}{dt}=1 \quad v=\sin t$
$= t\sin t - \int \sin t\,dt$

$y(t) = t\sin t - (-\cos t) + c$
$= t\sin t + \cos t + c$
$y(0) = 3$
$3 = \cos 0 + c$
$c = 2$

$y(t) = t\sin t + \cos t + 2$

d) $y(t) = t\cos t$
$t\cos t = 0$
$t \neq 0 \qquad \cos t = 0$
$t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} + 2 = \frac{\pi}{2} + 0 + 2 = \frac{\pi}{2} + 2$

3) $-8x^2 + 5xy + y^3 = -149$

a) $-16x + 5y + 5x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$
$\frac{dy}{dx}(5x + 3y^2) = 16x - 5y$
$\frac{dy}{dx} = \frac{16x - 5y}{5x + 3y^2}$

b) at $(4, -1)$ $\frac{dy}{dx} = \frac{16(4) - 5(-1)}{5(4) + 3(1)} = \frac{64 + 5}{23} = \frac{69}{23} = 3$

Equation of tangent:
$y - (-1) = 3(x - 4)$
$y + 1 = 3x - 12$
$y = 3x - 13$

c) $(4.2, k)$

$k = 3(4.2) - 13$
$k = -0.4$

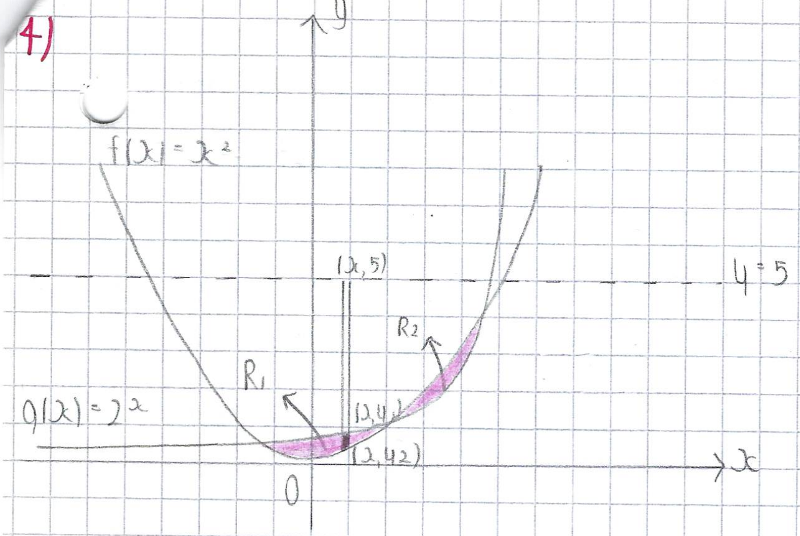
d) $-8(4.2)^2 + 5(4.2)(k) + k^3 = -149$
$-141.12 + 21k + k^3 = -149$
$k^3 + 21k + 7.88 = 0$

e) Using G.D.C.

$k = -0.373$ to 3.d.p

95: PART B

4)



a) $f(x)=g(x)$

$x^2=2^x$

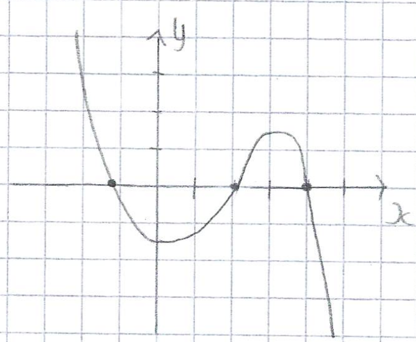
$x^2-2^x=0$

Using G.D.C.

$(-0.767, 0)$

$(2, 0)$

$(4, 0)$



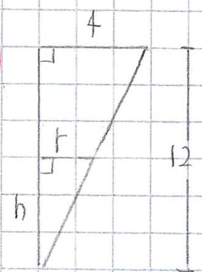
b) $\displaystyle\int_{-0.767}^{2}(2^x-x^2)\,dx+\int_{2}^{4}(x^2-2^x)\,dx$

c) $R(x)=5-y_2=5-x^2$

$r(x)=5-y_1=5-2^x$

Volume $=\pi\displaystyle\int_{-0.767}^{2}[(5-x^2)^2-(5-2^x)^2]\,dx$

5)



$\frac{dh}{dt}=h-12,\quad V=\frac{1}{3}\pi r^2 h$

a) $V=\frac{1}{3}\pi r^2 h$

Using similar triangles: $\frac{4}{r}=\frac{12}{h}$ $\qquad x=r$

$4h=12r$

$r=\frac{1}{3}h$

$V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h=\frac{1}{27}\pi h^3$

b) We want $\frac{dV}{dt}$ when $h=3$

$V = \frac{1}{27}\pi h^3$

$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

$= \frac{1}{9}\pi h^2 (h-12)$

$= \frac{1}{9}\pi (3^2)(-9)$

$= -9\pi$

-9π ft³/min

c) We want $\frac{dy}{dt}$ when $h=3$

$\pi R^2 = 400\pi$

$R^2 = 400$

$R = 20$

Volume of cylinder $= \pi R^2 y$

$\frac{dV}{dt} = 2\pi R y \frac{dR}{dt} + \pi R^2 \frac{dy}{dt}$

$\frac{dV}{dt} = \pi R^2 \frac{dy}{dt}$ $\quad \frac{dR}{dt} = 0$ since R is a constant

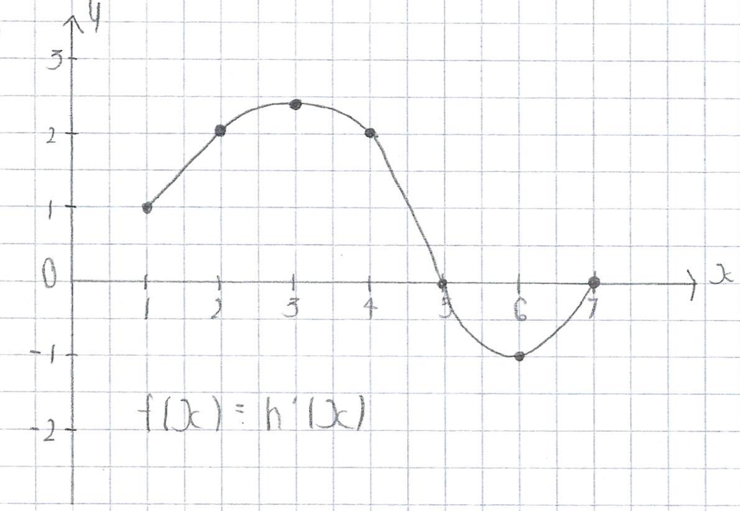
$9\pi = 400\pi \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{9}{400}$ ft/min

or: $V = 400\pi y$

$\frac{dV}{dt} = 400\pi \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{9\pi}{400\pi} = \frac{9}{400}$ ft/min

6) 

$h(x) = \int_1^x f(t)\,dt$

$f(x) = h'(x)$

a) $h(1) = \int_1^1 f(t)\,dt = 0$

b) $h'(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x)$

$h'(4) = f(4) = 2$

concave up on (1, 3) and (6, 7) since $h'(x)$ is increas[ing on]
those intervals

d) $h(x)$ has no relative minimum on $[1, 7]$ since $h'(x)$ does[n't]
change sign from -ve to +ve

$\therefore$ Minimum must occur at either endpoint

$h(1) = 0$

$h(7) = \int_1^7 f(t)\,dt$

$h(7) > h(1)$ since $h(x)$ has a relative maximum at $x = 5$ and cannot decrease to zero $\int_1^5 h'(x) > \int_5^7 h'(x)\,dx$